ON THE UNDERLYING LOGIC OF TWO SYSTEMS OF SET THEORY

BY

NEWTON C. A. DA COSTA

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1. *Introduction.*

This note is the third of a series devoted to the foundations of category theory. Our purpose is to study the underlying logic of T and T^* (cf. [3] and [4]), especially its completeness [1]).

The terminology, the notations, etc. are adapted from the works cited in the bibliography. In the metalanguage, we shall use the symbols $\Rightarrow$ (implies) and $\Leftrightarrow$ (equivalent).

2. *The logics L_2^* and L_2^{**}.*

In the formalization of T and T^* (cf. [4]), we have employed a two-sorted logic with identity (equality), which will be denoted by L_2^{**}, laying aside the non-constructive rule II_3 (and the constants V_1, V_2, ...). The corresponding two-sorted predicate calculus will be symbolized by L_2^*.

The notions of formula, of proof, etc. are the customary; however, we observe that each argument-place of a predicate symbol (predicate letter) may be filled by variables of both sorts. The first sort variables are $x, x', x'', \ldots, y, y', y'', \ldots, z, z', z'', \ldots$, and the variables of second sort are $t, t', t'', \ldots, u, u', u'', \ldots, v, v', v'', \ldots$. The primitive logical symbols are $\vee$ (or), $\neg$ (not) and V (for all); we define the abbreviations $\supset$ (implies), & (and), $\equiv$ (equivalent) and $\exists$ (there exists) as it is usual.

Postulates of L_2^*:

A, B and C are formulas.

$$\mathrm{I}_1 \quad A \vee A \supset A \qquad \mathrm{I}_2 \quad A \supset A \vee B \qquad \mathrm{I}_3 \quad A \vee B \supset B \vee A$$

$$\mathrm{I}_4 \; (A \supset B) \supset (C \vee A \supset C \vee B) \qquad \mathrm{I}_5 \; \frac{A \quad A \supset B}{B}$$

If α is a variable, $A(\alpha)$ is a formula and C is a formula which does not contain α free, then:

$$\mathrm{II}_1 \; \frac{C \supset A(\alpha)}{C \supset V\alpha\, A(\alpha)}$$

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$A(\alpha)$ is a formula, β is a variable free for α in $A(\alpha)$ and if α is of second sort, β must be of second sort:

$$\mathrm{II}_2 \quad \forall \alpha A(\alpha) \supset A(\beta)$$

Postulates of L_2^{**}:

The postulates of L_2^{**} are I_1–II_2 above and the following:

$$\mathrm{III}_1 \quad \forall x(x=x)$$

$A(\alpha)$ is a formula and β and γ are distinct variables free for α in $A(\alpha)$:

$$\mathrm{III}_2 \quad \beta=\gamma \supset (A(\beta) \equiv A(\gamma)).$$

If the schema II_2 is formulated in the usual form, that is, if it is required that α and β must be variables of the same sort, then L_2^{**} becomes the usual two-sorted predicate calculus with equality, $L_2^{=}$, and L_2^{*} becomes the customary two-sorted predicate calculus, L_2.

Theorem 1. If $\vdash F$ in L_2^{*}, then F is valid in all sets of two non-empty domains S_1 and S_2, $S_2 \subset S_1$, such that S_1 and S_2 are the ranges of the variables of first and second sorts.

Theorem 2. L_2^{*} is complete: every formula F, valid in all sets of two non-empty domains, S_1 and S_2, under the conditions of the preceding theorem, is provable in L_2^{*}.

Proof. Any of the standard completeness proofs for the two-sorted predicate calculus, L_2, may be adapted to demonstrate the completeness of L_2^{*}.

Lemma 1. In L_2^{**} (x and t are variables of first and second sorts): $\vdash \forall t \exists x(x=t)$.

Theorem 3. If $\vdash F$ in L_2^{**}, then F is valid in all sets of two non-empty domains S_1 and S_2, $S_2 \subset S_1$ (S_1 and S_2 are respectively the ranges of the variables of first and second sorts; $=$ is interpreted as the relation of equality).

Theorem 4. L_2^{**} is complete: every formula valid in all sets of two non-empty domains S_1 and S_2, under the conditions of theorem 3, is provable in L_2^{**}.

As usual, in the calculi L_2^{*} and L_2^{**}, $\Gamma \models F$ means that F is a *semantic consequence* of (the formulas) Γ.

Theorem 5. Let $\Gamma \vdash F$ be a deduction of L_2^{*} (or L_2^{**}) without variation of variables. Then, we have in L_2^{*} (or in L_2^{**}): $\Gamma \models F \Leftrightarrow \Gamma \vdash F$.

It hardly needs saying that the Herbrand-Schmidt theorem, for instance in the form of SMILEY [9], p. 67, can be conveniently reformulated to cover L_2^{*} and L_2^{**}. Clearly, it is possible to generalize the foregoing considerations in several ways (for example, imposing other restrictions on S_1 and S_2 or taking into account more than two sorts of variables).

Let L_n and $L_n^=$, $1 \leqslant n \leqslant \omega$, be respectively the n-sorted predicate calculus and the n-sorted predicate calculus with equality. If we generalize the notion of P-k-transform of a formula of L_1 and adapt the exposition of [1], we may prove, for instance, the following results (the proofs are finitary in the sense of KLEENE [8]):

Theorem 6. F is a formula which does not contain the symbol $=$. Then, $\vdash F$ in $L_2{}^*$ (or L_n) if and only if $\vdash F$ in $L_2{}^{**}$ (or $L_n^=$, $1 \leqslant n \leqslant \omega$).

Theorem 7. For every quantifier free formula of $L_2{}^*$ (or L_n, $1 \leqslant n \leqslant \omega$), there is a proof in which only quantifier-free formulas occur.

Theorem 8. If the formula F does not contain the symbol $=$, $\vdash F$ in the n-sorted intuitionistic (minimal) predicate calculus if, and only if, $\vdash F$ in the n-sorted intuitionistic (minimal) predicate calculus with equality. This is also true for the implicative and positive calculi.

It is plain that results similar to the preceding ones can be established for the one-sorted predicate calculus with restricted variables (cf. [5] and [6]). Generally speaking, the true results for L_n (or $L_n^=$), $1 \leqslant n \leqslant \omega$, with a few modifications, are also valid for L_1 (or $L_1^=$) with restricted variables and conversely. In particular, it is possible to formalize T and T^* using as underlying logic a strengthened form of $L_1^=$ and the cited species of variables.

There is no difficulty to introduce the operation of restriction of variables in $L_2{}^*$, $L_2{}^{**}$, L_n and $L_n^=$, $1 \leqslant n \leqslant \omega$, and this operation has the expected properties.

3. *α-completeness.*

To simplify, we assume that: 1) α and β are ordinal numbers of Von Neumann; 2) if $\alpha \geqslant \omega$, then $0 < \beta \leqslant \alpha$; 3) if $\alpha < \omega$, then $0 < \beta \leqslant \omega$.

Let C_α be the classical predicate calculus with the family $(c_i)_{i \in \alpha}$ of (distinct) constants. The postulates of C_α are, for instance, I_1–II_2 of [4], conveniently adapted to take into account the constants c_i, $i \in \alpha$, and the fact that there is only one sort of variables. We shall denote by $C_\alpha{}^*$ the calculus C_α to which we have added the following rule [2]):

$$(\alpha) \quad \frac{A(c_0), A(c_1), A(c_2), \ldots}{V x A(x)},$$

where $A(x)$ is a formula and $A(c_0)$, $A(c_1)$, $A(c_2)$, ... are obtained from $A(x)$ by substituting $c_0, c_1, c_2, \ldots$ for x.

The (distinct) predicate symbols of $C_\alpha{}^*$ are R_j, $j \in \beta$, and a formula

[2]) A rule of this type was first considered by Carnap [Ein Gültigkeitskriterium für die Sätze der klassischen Mathematik, Monatshefte für Math. und Physik, 42, 163–190 (1935)].

of this calculus with no free variables is called a statement. In the sequel, Σ and E will denote respectively a set of statements and a statement.

Definition I. Let Γ be a set of formulas of C_α^*. A formula F is said to be associated with Γ, if F is an immediate consequence of formulas of Γ by rule (α).

Definition II. A formula F is a (syntactic) α-consequence of a set of formulas Γ, and we write $\Gamma \vdash_{\alpha} F$, if and only if, there exists a finite sequence of formulas, $D_1, D_2, \ldots, D_n$, where D_n is F, such that, for each i, $1 \leqslant i \leqslant n$, we have: a) D_i is an axiom of C_α^*, or b) D_i is an immediate consequence of preceding formulas by one of the rules of C_α^* (except (α)), or c) D_i is a formula associated with Γ or with a set of α-consequences of Γ. The sequence $D_1, D_2, \ldots, D_n$ is called an α-deduction of F from (the hypotheses) Γ. If $\Gamma = \emptyset$, the α-deduction is an α-proof and F is said to be an α-theorem.

Remark. Suitable forms of the usual metatheorems of C_α (the deduction theorem, the rule of proof by cases, . . .) are true for C_α^*.

Definition III. Let M be a set such that $0 < \overline{\overline{M}} \leqslant \overline{\overline{\alpha}}$. An α-interpretation I_M of C_α^* is an interpretation of the constants and predicate symbols of C_α^* in M (cf. Cohen [2], p. 12), satisfying the condition: to each element $m \in M$ there exists at least a constant which is its "name" according to I_M.

Definition IV. An α-model of a set of statements Γ of C_α^* is an α-interpretation in which all statements of Γ are true (cf. Cohen [2], pp. 12–13).

Definition V. In C_α^*, E is a semantic α-consequence of Σ if every α-model of Σ is also an α-model of $\{E\}$. In this case, we write $\Sigma \models_{\alpha} E$.

$\widetilde{\Sigma}$ symbolizes the set of all statements associated with Σ (including the elements of Σ itself). $\bar{\Sigma}$ is the set of all syntactic α-consequences of Σ.

Lemma 2. In C_α^*, if Σ is consistent, then Σ has an α-model.

Proof. For instance, by a (non trivial) modification of the proof of Gödel's generalized theorem, also called Gödel-Malcev-Henkin theorem, contained in Cohen [2], pp. 13–16.

Lemma 3. In C_α^*, if Σ is consistent, then $\widetilde{\Sigma}$ is consistent.

Proof. If Σ is consistent, Σ has an α-model M. But all statements associated with Σ are also true in M. Therefore, $\widetilde{\Sigma}$ has an α-model and is consistent.

Lemma 4. In C_α^*: Σ is consistent $\Rightarrow$ $\bar{\Sigma}$ is consistent.

We have, evidently, the following theorem:

Theorem 9. $\Sigma \vdash_{\alpha} E \Rightarrow \Sigma \models_{\alpha} E$.

Theorem 10. $\Sigma \models_{\alpha} E \Rightarrow \Sigma \vdash_{\alpha} E$.

Proof. If we have not that $\Sigma \vdash_{\alpha} E$, the set $\Sigma \cup \{\neg E\}$ would be consistent and would have an α-model. Then, we should not have $\Sigma \models_{\alpha} E$ and this is absurd.

Supposing α finite, that is, supposing that α is a natural number greater than 0, we have the following proposition ($\alpha = n \in \omega$ and $\alpha \neq 0$):

Theorem 11. $\Sigma \models_{n} E$ if, and only if, $\Sigma \vdash_{n} E$ or, in other words, if we can deduce E from Σ in C_n^*, using the rule:

$$(n) \quad \frac{A(c_0) \,\&\, A(c_1) \,\&\, \ldots \,\&\, A(c_{n-1})}{\forall x A(x)} \; ^{3)}.$$

Theorem 12. $\Sigma \vdash_{\alpha} E \Leftrightarrow \Sigma \models_{\alpha} E$.

Theorem 13. If $\alpha \geqslant \omega$, E is a thesis of C_α if and only if E is an α-theorem.

The above results are true for $\mathscr{I}_\alpha$, the so called predicate calculus with equality and the family $(c_i)_{i \in \alpha}$ of constants. We denote by $\mathscr{I}_\alpha^*$ the calculus obtained from $\mathscr{I}_\alpha$ by adjoining rule (α).

The preceding considerations show that C_α^* and $\mathscr{I}_\alpha^*$ are α-complete: $\models_{\alpha} E \Rightarrow \vdash_{\alpha} E$. In general, if E is a semantic α-consequence of Σ, then E is also a syntactic α-consequence of Σ.

4. *The logics $\mathscr{L}$ and $\mathscr{L}^=$.*

$\mathscr{L}(\mathscr{L}^=)$ will denote the predicate calculus (the predicate calculus with equality) of T and T^*. (The postulates of $\mathscr{L}$ and $\mathscr{L}^=$ are respectively I_1–II_3 and I_1–III_2 of [4].)

It is a consequence of the foregoing exposition that $\mathscr{L}$ and $\mathscr{L}^=$ are complete:

Theorem 14. In $\mathscr{L}$ or $\mathscr{L}^=$: $\Sigma \vdash E$ if, and only if, $\Sigma \models E$. (The meanings of $\vdash$ and $\models$ are clear.)

Corollary 1. In $\mathscr{L}$ or $\mathscr{L}^=$: $\models E \Leftrightarrow \vdash E$.

Corollary 2. If F is a formula which does not contain the symbol of equality, then $\vdash F$ in $\mathscr{L}^=$ if, and only if, $\vdash F$ in $\mathscr{L}$ [4]).

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[3]) This well-known theorem forms part of the proof that the problem of validity (or satisfiability) in domains with n ($0 < n \in \omega$) elements is decidable, via the method of elimination of quantifiers.

[4]) Evidently, our results on α-completeness are related to certain ideas of Prof. Henkin [A generalization of the concept of ω-consistency, The Journal of Symb,

5. *Generalized formulas.*

The systems T and T^* are easy to handle, especially if we introduce the notion of a generalized formula. We shall treat this notion only in $C_\alpha{}^*$; for the many-sorted predicate calculus (with or without equality), $\mathscr{L}$ and $\mathscr{L}^=$, the concept of generalized formula may be extended and has the espected properties.

We call the symbols c_i, c_j, c_k, c_{i1}, c_{j1}, c_{k1}, c_{i2}, c_{j2}, c_{k2}, ... generalized variables. If in a formula F of $C_\alpha{}^*$ we substitute a generalized variable for the free occurrences of a variable, according to the customary conventions, the resulting expression F^* is said to be a generalized formula. The common formulas are particular instances of generalized formulas and in these last formulas a generalized variable may occur only free. A generalized statement is a generalized formula with no free occurrences of usual variables.

[For example, if in the formula ($\in$ is a binary predicate symbol) $Vx(x \in y \vee \neg (y \in z))$, we substitute c_i and c_j respectively for y and z, we obtain the generalized formula $Vx(x \in c_i \vee \neg (c_i \in c_j))$.]

In $C_\alpha{}^*$, the meaning of

$$\vdash_\alpha A(c_i)$$

is that one has

$$\vdash_\alpha A(c_0), \qquad \vdash_\alpha A(c_1), \qquad \vdash_\alpha A(c_2), \ldots$$

Similarly, the meaning of the expression

$$\vdash_\alpha A(c_i, c_j)$$

is well defined.

The rule (α) may, for example, be formulated as follows:

$$(\alpha)' \quad \frac{A(c_i)}{VxA(x)},$$

with clear restrictions.

The notions of α-deduction, α-interpretation, α-model, etc., may be extended to cover the generalized formulas. Hence, if $\Gamma \cup \{F\}$ is a set of generalized formulas, $\Gamma \vdash_\alpha F$ and $\Gamma \models_\alpha F$ are meaningful.

For example, the following propositions are true:

Theorem 15. In $C_\alpha{}^*$, if $\Gamma \cup \{F\}$ is a set of generalized statements, we have: $\Gamma \vdash_\alpha F \Leftrightarrow \Gamma \models_\alpha F$.

Logic, **19**, 183–196 (1954), and A generalization of the concept of ω-completeness, The Journal of Symb. Logic, 22, 1–14 (1957)]. In a future paper, we intend to analyse such relations and the question to what extent our results can be extended to a wide class of infinitary languages. (Prof. Henkin has called our attention to these problems.)

Theorem 16. Under the conditions of the preceding theorem, we have in $\mathscr{I}_\alpha{}^*$: $\Gamma \vdash_{\alpha} F \Leftrightarrow \Gamma \models_{\alpha} F$.

In the case of T and T^*, the constants are $V_1, V_2, V_3, \ldots$ and the generalized variables V_m, V_n, V_p, etc. (cf. [3] and [4]).

6. *Corrections to* [4].

1) In Definition I, instead of "If α and β are terms and γ is a variable of first sort distinct of α and β, ", read "If α and β are terms and γ is a variable of first sort distinct from the free variables of α and β, "; some other definitions of that paper must suffer similar corrections.

2) Page 46, instead of "The definitions of (formal) proof, of (formal) deduction, etc., are as in KLEENE [2].", read "The definitions of (formal) proof, of (formal) deduction, etc., are as in KLEENE [2], conveniently modified.".

3) In his review of [4], *MR* 35 # 65941 (1968), Prof. G. Asser wrote: "Bemerkungen des Referenten: Leider finden sich in der Arbeit einige grundlegende Fehler: (1) Die Behauptung des Autors, man könne die von ihm benutzte Regel der unendlichen Induktion eliminieren, ist auf grund bekannter Resultate der Metamathematik falsch. (2) In der Arbeit findet sich die folgende "Definition": Ein Term (!) α heisst von Typ n, $n = 1, 2, 3, \ldots$, wenn $\alpha \in V_n$ (V_n ist dabei eine Konstante des Formalismus). Was soll hierbei "$\alpha \in V_n$" bedeuten? Bedeutet es "$\vdash \alpha \in V_n$", was wohl eigentlich nur infrage käme, so ist das für den Fall, dass α eine Variable ist, sicher falsch. Dann wird aber die nachfolgende Definition für "α ist ein Universum" in diesen Fall sinnlos. Sinnlos werden dann aber auch Theoreme wie "$\forall x \exists y$ ($x \in y$ & y ist ein Universum)", an denen dem Autor im Hinblick auf sein Anliegen, eine exakte Grundlage für die Theorie der Kategorien zu schaffen, besonders gelegen ist."

We must say the following. (a) With reference to (1): We have not employed the verb "to eliminate" in the metamathematical precise sense and we thought this was clear by the context. It is possible *to eliminate* rule II_3 of [4], but instead we must modify profoundly T and T^*; for example, axiomatizing the properties of the sequence of universes $V_1, V_2, V_3, \ldots$, and the new systems are evidently not equivalent to the former ones. (b) With reference to (2): The definition of term of type n, $n = 1, 2, \ldots$ is not rigorously formulated; notwithstanding, its meaning seems to us evident. The "good" contextual definition (or family of definitions) is:

If α is a term,

$$\alpha \text{ is of type } n \equiv \alpha \in V_n \qquad (n = 1, 2, 3, \ldots).$$

(V_n is not, as Prof. Asser wrote, a constant of the formalized system, but simply a metamathematical variable.)

7. *Remark added in proof.*

Our results on α-completeness ($\alpha \geqslant \omega$), and their consequenses of sections 4 and 5 of this note, seem to be true only if we adjoin to $C_\alpha{}^*$ ($I_\alpha{}^*$) the following rule ("dual" of rule (α)):

$$\frac{\exists x A(x)}{A(c_0) \text{ or } A(c_1) \text{ or} \ldots},$$

or other equivalent postulate.

Institute of Mathematics
University of Campinas
Campinas, São Paulo, Brazil

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